Monodromy of Compositions of Toroidal Belyĭ Maps

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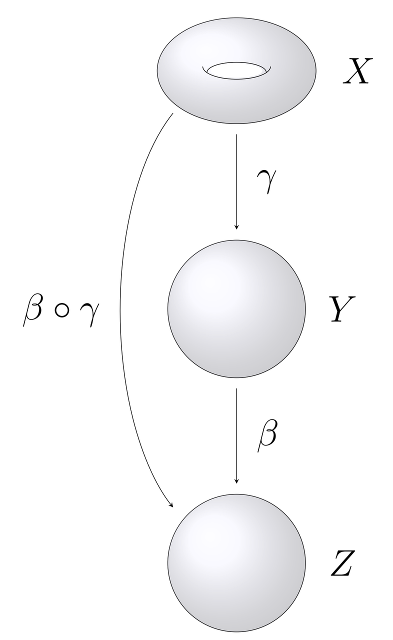
Abstract

Say that $\beta : \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is a Dynamical Belyĭ map. Given any Toroidal Belyĭ map $\gamma : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$, the composition $\beta \circ \gamma : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ is also a Toroidal Belyĭ map. There is a group $\mathrm{Mon}(\beta)$, the monodromy group, which contains information about the symmetries of a Belyĭ ı map β. It is well-known that, for any Toroidal Belyĭ ı map γ, (i) there is always a surjective group homomor- phism $\mathrm{Mon}(\beta \circ \gamma) \twoheadrightarrow \mathrm{Mon}(\beta)$, and (ii) the monodromy group $\mathrm{Mon}(\beta \circ \gamma)$ is contained in the $\mathrm{Mon}(\gamma) \wr \mathrm{Mon}(\beta)$.

In this project, we study how the three groups $\mathrm{Mon}(\beta)$, $\mathrm{Mon}(\beta \circ \gamma)$, and $\mathrm{Mon}(\gamma) \wr \mathrm{Mon}(\beta)$ compare as we vary over Dynamical Belyĭ maps β and now Toroidal Belyĭ maps γ. This is work done as part of the Pomona Research in Mathematics Experience (NSA H98230-21-1-0015).

Toroidal Belyi Map

A **Toroidal Belyĭ map** is a mapping $\gamma : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ from an elliptic curve E to a Riemann sphere. A **Toroidal Belyĭ pair** is (E, γ). $\beta \circ \gamma : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$



Jacob Bond's Theorems

Corollary (pg. 71)

The monodromy group $\mathrm{Mon}(\beta\gamma)$ of the composition of a dynamical Belyĭ map β and a Belyĭ map γ is isomorphic to a subgroup of the wreath product $\mathrm{Mon}(\gamma) \wr_{E_\beta} \mathrm{Mon}(\beta)$. Moreover, this isomorphism is given by

$$\begin{aligned} \mathrm{Mon}(\beta\gamma) &\to \varphi_\gamma(\pi_1^Z) \le \mathrm{Mon}(\gamma) \wr_{E_\beta} \mathrm{Mon}(\beta) \\ \rho_{\beta\gamma}(\lambda) &\mapsto (\rho_{\gamma\star}(f_\lambda), \rho_\beta(\lambda)) \end{aligned}$$

Theorem 4.18 (pg. 76)

Let β be a dynamical Belyĭ map with constellation (τ_0, τ_1), and extending pattern (f_0, f_1). Let

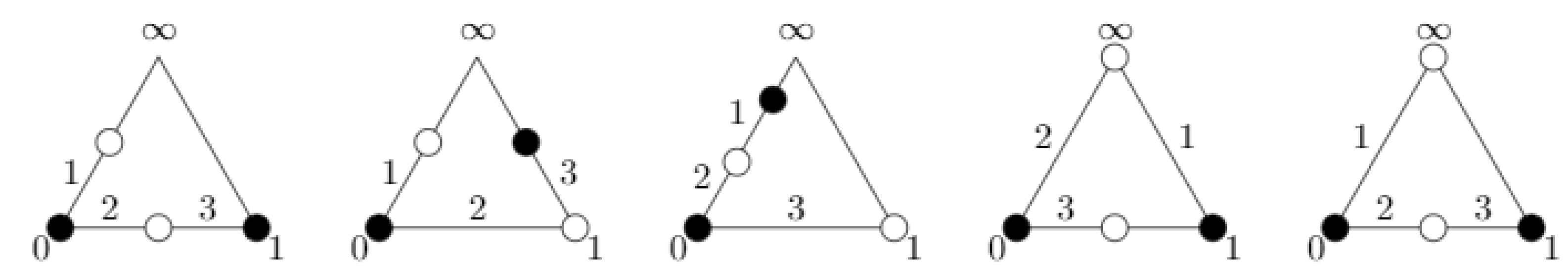
$$\varphi : \begin{array}{l} g_0 \mapsto (f_0, \tau_0) \\ g_1 \mapsto (f_1, \tau_1) \end{array}$$

and $A := \varphi(\mathrm{Ker}\rho_\beta)$. Then for any Belyĭ map γ,

$$\mathrm{Mon}(\beta\gamma) \cong \rho_{\gamma\star}(A) \rtimes \mathrm{Mon}(\beta)$$

Extending Pattern Examples (Melanie Wood)

i	$\beta(t)$	Extending Pattern	Generators	Sufficient conditions for $\mathrm{Mon}(\beta_i\gamma) \cong \mathrm{Mon}(\gamma) \wr \mathrm{Mon}(\beta_i)$
1	$-\frac{27}{4}(t^3 - t^2)$	$\tau_0 = (12)$ $f_0 = [a, 1, b]$ $\tau_1 = (23)$ $f_1 = [1, 1, 1]$	$[a^{-2}, b^{-1}, b^{-1}], [1, 1, 1], [b^{-1}, a^{-2}, b^{-1}],$ $[ab^{-1}a^{-1}, b^{-1}, ba^{-2}b^{-1}],$ $[a^{-1}, ab^{-1}, ba^{-1}b^{-1}]$	$\mathrm{Mon}(\gamma) = \langle a_\gamma^2 \rangle$ or $a_\gamma = 1$ (so that $\mathrm{Mon}(\gamma) = \langle b_\gamma \rangle$)
2	$-2t^3 + 3t^2$	$\tau_0 = (12)$ $f_0 = [a, 1, 1]$ $\tau_1 = (23)$ $f_1 = [1, b, 1]$	$[a^{-1}, a^{-1}, 1], [1, b^{-1}, b^{-1}], [ab^{-1}a^{-1}, 1, b^{-1}],$ $[a^{-1}, 1, a^{-1}], [1, a^{-1}, a^{-1}],$ $[1, ba^{-1}, 1], [ab^{-1}a^{-1}, a^{-1}, 1]$	$\mathrm{Mon}(\gamma) = \langle a_\gamma^2 \rangle$ or $\mathrm{Mon}(\gamma) = \langle b_\gamma^2 \rangle$
3	$\frac{t^3+3t^2}{4}$	$\tau_0 = (23)$ $f_0 = [1, a, 1]$ $\tau_1 = (12)$ $f_1 = [1, 1, b]$	$[1, a^{-1}, a^{-1}], [1, 1, b^{-2}], [a, ab^{-2}a^{-1}, 1],$ $[a^{-1}, 1, ba^{-1}b^{-1}], [a^{-1}, aba^{-1}b^{-1}a^{-1}, 1],$ $[aba^{-1}, b^{-1}a^{-1}, b], [ab^{-1}a^{-1}, b^{-1}a^{-1}, b]$	$\mathrm{Mon}(\gamma) = \langle a_\gamma^2 \rangle$ or $\mathrm{Mon}(\gamma) = \langle b_\gamma^2 \rangle$
4	$\frac{27t^2(t-1)}{(3t-1)^3}$	$\tau_0 = (23)$ $f_0 = [b, a, 1]$ $\tau_1 = (12)$ $f_1 = [b^{-1}a^{-1}, 1, 1]$	$[b^{-2}, a^{-1}, a^{-1}], [ab, ab, 1], [ba, 1, ab],$ $[b^{-1}a^{-1}b, b^{-2}, a^{-1}], [a^{-1}, a^{-1}, b^{-2}],$ $[b^{-1}, a^{-2}, b^{-1}], [b^{-1}, ba^{-1}, a]$	$\mathrm{Mon}(\gamma) = \langle c_\gamma^2 \rangle$
5	$\frac{t^2(t-1)}{(t-\frac{4}{3})^3}$	$\tau_0 = (12)$ $f_0 = [a, 1, b]$ $\tau_1 = (23)$ $f_1 = [b^{-1}a^{-1}, 1, 1]$	$[a^{-1}, a^{-1}, b^{-2}], [b^{-1}a^{-1}b, b^{-2}, a^{-1}],$ $[abab, 1, 1], [1, abab, 1],$ $[ab^{-2}a^{-1}, b^{-1}a^{-1}b, ba^{-1}b^{-1}],$ $[b^{-2}a^{-1}, b^2, b^{-1}a^{-1}b^{-1}], [b^{-2}a^{-1}, b^2, a]$	$\mathrm{Mon}(\gamma) = \langle c_\gamma^2 \rangle$



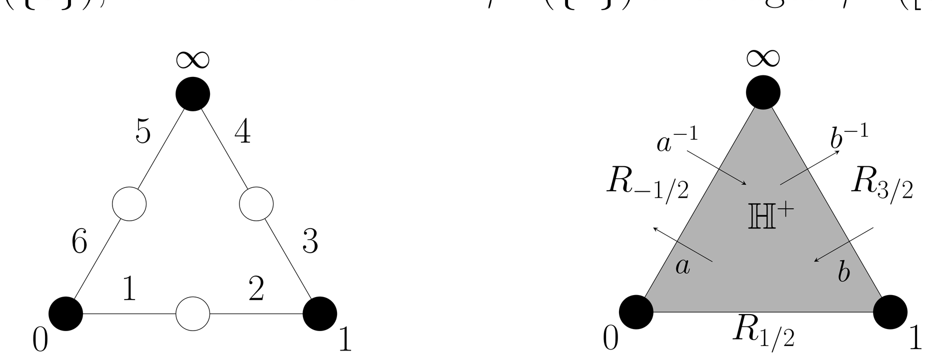
β_1 β_2 β_3 β_4 β_5

Motivating Question

When is $\mathrm{Mon}(\beta \circ \gamma)$ **equal** to $\mathrm{Mon}(\gamma) \wr \mathrm{Mon}(\beta)$?

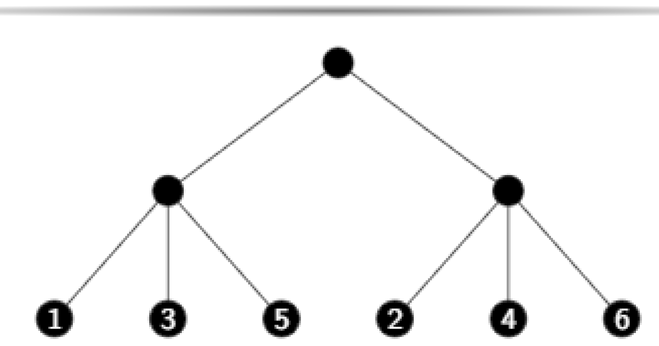
Dessin d'Enfants & Extending Patterns

Given a Belyĭ pair (X, γ) we define the **Dessin d'Enfant** as the bipartite graph embedded in X with black vertices $B = \gamma^{-1}(\{0\})$, white vertices $W = \gamma^{-1}(\{1\})$ and edges $\gamma^{-1}([0, 1])$.



$\tau_0 = (1, 6)(2, 3)(4, 5)$ $f_0 = [1, b, 1, b^{-1}a^{-1}, 1, a]$
$\tau_1 = (1, 2)(3, 4)(5, 6)$ $f_1 = [1, 1, 1, 1, 1, 1]$

Wreath Product



Permute the 'arms' and 'hands' of the mobile:

$$G \wr H = G^n \rtimes H \text{ and } |G \wr H| = |G|^n |H|$$

Results

Proposition 1.

The monodromy group, $\mathrm{Mon}(\beta) = \langle \tau_0, \tau_1 \rangle$ where $\tau_0 = (1, 2, \ldots, n)$, $\tau_1 = \mathrm{id}$, and $\mathrm{Mon}(\beta) = C_n$. We also have $f_0 = (1, \ldots, 1, a, 1, \ldots, 1)$ (where a is in entry $\lfloor \frac{n}{2} \rfloor + 1$ of f_0) and $f_1 = (b, 1, \ldots, 1)$.

Proposition 2.

$\mathrm{Ker}(\rho_\beta) = \langle b, a^n, a^i b a^{-i} \rangle$ for $i \in \{\pm 1, \ldots, \pm \lfloor \frac{n}{2} \rfloor\}$.

Proposition 3.

$\rho_{\gamma^*}(A) = \langle (b_\gamma, 1, \ldots, 1), \ldots, (1, \ldots, 1, b_\gamma), (a_\gamma, \ldots, a_\gamma) \rangle$ where b_γ appears in each of n positions.

Lemma

Given the homomorphism φ such that

$$\varphi(a) = [f_0, \tau_0]$$
$$\varphi(b) = [f_1, \tau_1]$$

we obtain the following relations:

$$\begin{aligned} \varphi(b) &= [(b, 1, \ldots, 1); \mathrm{id}] \\ \varphi(a^n) &= [(a, \ldots, a); \mathrm{id}] \\ \varphi(a^i b a^{-i}) &= [(1, \ldots, 1, d, 1, \ldots, 1); \mathrm{id}]. \end{aligned}$$

Here, d is in the i^{th} position of $\varphi(a^i b a^{-i})$ and

$$d = \begin{cases} aba^{-1} & \text{if } |i| = \lfloor \frac{n}{2} \rfloor \\ b & \text{otherwise} \end{cases}$$

Theorem

Theorem

Let $\mathrm{Mon}(\gamma) = \langle a_\gamma, b_\gamma \rangle$ be abelian and $\beta(z) = z^n$ for some $n > 1$. Then $\mathrm{Mon}(\beta\gamma) \cong \mathrm{Mon}(\gamma) \wr \mathrm{Mon}(\beta)$ if and only if $\mathrm{Mon}(\gamma) = \langle b_\gamma \rangle$.

Proof Sketch

1. Calculate τ_0, τ_1 and f_0, f_1 for β.
2. Determine generators of $\mathrm{Ker}(\rho_\beta)$.
3. Find generators of $A := \varphi(\mathrm{Ker}(\rho_\beta))$ and subsequently, $\rho_{\gamma^*}(A)$.
4. Show $\mathrm{Mon}(\gamma) = \langle b_\gamma \rangle$ implies $\rho_{\gamma^*}(A) \cong (\mathrm{Mon}(\gamma))^n$.
5. Show $\rho_{\gamma^*}(A) \cong (\mathrm{Mon}(\gamma))^n$ implies $\mathrm{Mon}(\gamma) = \langle b_\gamma \rangle$.

Future Work

- Investigating case where $\mathrm{Mon}(\gamma)$ is non-abelian
- Considering other compositions, for example $E(\mathbb{C}) \to E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ or involving surfaces of genus > 1

References

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